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Carmeli's Gravitational Field Equations

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Carmeli has proposed spinorial field equations in curved space-time to describe gravitation. In this paper we give the relationship between these equations and the standard Einstein gravitational field equations. In particular we show that all solutions to Einstein's equations are solutions to Carmeli's equations, but not vice versa.

KEY WORDS: Carmeli; spinorial field equations; gravitational field equations.

1. INTRODUCTION

In his book (Carmeli, 1982), Carmeli has proposed the following spinorial field equations in curved space-time to describe gravitation:

$$\nabla_\nu F_{AB}{}^{\mu\nu} = J_{AB}{}^{\mu}, \tag{1a}$$

$$\nabla_\nu {}^{*}F_{AB}{}^{\mu\nu} = 0, \tag{1b}$$

where $F_{AB}{}^{\mu\nu}$ is the curvature spinor, and ${}^{*}F_{AB}{}^{\alpha\beta}$ is its dual, both to be defined in the sequel. The "current density" $J_{AB}{}^{\mu}$ was not specified by him. The elegance of these equations can be readily seen as they are similar to Maxwell's equations:

$$\nabla_\nu F^{\mu\nu} = J^{\mu}, \tag{2a}$$

$$\nabla_\nu {}^{*}F^{\mu\nu} = 0, \tag{2b}$$

where $F^{\mu\nu}$ is the electromagnetic field. It will be noticed that Eqs. (1) are also esthetically identical to Yang–Mills gauge field equations. The gauge aspects of Eqs. (1) will be shown elsewhere.

It is the purpose of this paper to discuss in detail some aspects of these equations, which will be referred to henceforth as Carmeli's equations. It will be shown that the Einstein field equations and these equations are closely related. In fact, for a proper choice of $J_{AB}{}^{\alpha}$ any solution to Einstein's equations will satisfy (1a) and (2b) identically.

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Our discussion will be done using spinor calculus, which will be introduced in brevity. For more details the reader is referred to Carmeli (1977, 1982).

2. PRELIMINARIES

This section is a brief introduction to the formalism that will be used in this paper and will fix our notation henceforth.

2.1. Spinor Representation of *SL*(2,C)

Let G_{mn} be the set of all polynomials $p(z, \bar{z})$ of degree no larger than m in z and n in its complex conjugate $\bar{z}$. Thus under the law of polynomial addition and multiplication by a scalar, G_{mn} constitutes a finite $(m+1) \times (n+1)$-dimensional vector space.

Let $g \in SL(2,C)$ be the group of all 2×2 complex matrices with unit determinant, and define the linear operator $T_g : G_{mn} \to G_{mn}$ such that

$$T_g \circ p(z, \bar{z}) \equiv (g_{01}z + g_{11})^m (\bar{g}_{01}\bar{z} + \bar{g}_{11})^n \, p\left(\frac{g_{00}z + g_{10}}{g_{01}z + g_{11}}, \frac{\bar{g}_{00}\bar{z} + \bar{g}_{10}}{\bar{g}_{01}\bar{z} + \bar{g}_{11}}\right). \quad (2.1)$$

In the above equation g_{lk}, where $l, k \in \{1, 2\}$, are elements of the matrix $g \in SL(2,C)$:

$$g = \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix}. \quad (2.2)$$

It can be verified by substitution that $T_g T_h = T_{gh}$ for all g, h in $SL(2,C)$. The correspondence $g \to T_g$ is thus a finite–dimensional representation of $SL(2,C)$.

Now let $\tilde{G}_{mn}$ be the set of all complex numbers $\psi_{A_1 \cdots A_m B'_1 \cdots B'_n}$ which are symmetric in the indices $A_1, \ldots, A_m$ and in $B'_1, \ldots, B'_n$ where all the indices take the values 0 and 1. Under the normal law of addition and multiplication by a scalar it is thus clear that $\tilde{G}_{mn}$ constitutes another $(m+1) \times (n+1)$-dimensional vector space.

Now for each $\psi_{A_1 \cdots A_m B'_1 \cdots B'_n}$ in $\tilde{G}_{mn}$ we can define $p(z, \bar{z}) \in G_{mn}$ such that

$$p(z, \bar{z}) = \psi_{A_1 \cdots A_m B'_1 \cdots B'_n} z^{A_1 + \cdots + A_m} \bar{z}^{B'_1 + \cdots + B'_n}, \quad (2.3)$$

where the summation convention is used. Conversely, if $p(z, \bar{z}) \in G_{mn}$, then there exist coefficients a_{lk} such that

$$p(z, \bar{z}) = a_{lk} z^l \bar{z}^k. \quad (2.4)$$

To each such coefficient we can define uniquely an element $\psi_{A_1 \cdots A_m B'_1 \cdots B'_n}$ of $\tilde{G}_{mn}$ by

$$\psi_{A_1 \cdots A_m B'_1 \cdots B'_n} = \binom{m}{A_1 + \cdots + A_m} \binom{n}{B'_1 + \cdots + B'_n} a_{A_1 + \cdots + A_m \, B'_1 + \cdots + B'_n} \quad (2.5)$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

We thus have a one-to-one correspondence between $\tilde{G}_{mn}$ and G_{mn}. There must therefore be a one-to-one correspondence between linear operators on $\tilde{G}_{mn}$ and linear operators on G_{mn}. Let $\tilde{T}$ denote the operator on $\tilde{G}_{mn}$, which corresponds to the operator T on G_{mn}

$$\begin{aligned} T_g \circ p(z, \bar{z}) &= T_g \circ \left(\psi_{A_1\cdots A_m B'_1\cdots B'_n} z^{A_1+\cdots+A_m} \bar{z}^{B'_1+\cdots+B'_n}\right) \\ &= (g_{01}z + g_{11})^m (\bar{g}_{01}\bar{z} + \bar{g}_{11})^n \psi_{A_1\cdots A_m B'_1\cdots B'_n} \\ &\quad \times \left(\frac{g_{00}z + g_{10}}{g_{01}z + g_{11}}\right)^{A_1+\cdots+A_m} \left(\frac{\bar{g}_{00}\bar{z} + \bar{g}_{10}}{\bar{g}_{01}\bar{z} + \bar{g}_{11}}\right)^{B'_1+\cdots+B'_n} \\ &= \left(\tilde{T}_g \circ \psi_{A_1\cdots A_m B'_1\cdots B'_n}\right) z^{A_1+\cdots+A_m} \bar{z}^{B'_1+\cdots+B'_n}. \end{aligned} \tag{2.6}$$

A long but straightforward calculation yields

$$\begin{aligned} \psi'_{A_1\cdots A_m B'_1\cdots B'_n} &\equiv \tilde{T}_g \circ \psi_{A_1\cdots A_m B'_1\cdots B'_n} \\ &= g_{A_1}{}^{C_1} \cdots g_{A_m}{}^{C_m} \bar{g}_{B'_1}{}^{D'_1} \cdots \bar{g}_{B'_n}{}^{D'_n} \psi_{C_1\cdots C_m D'_1\cdots D'_n} \end{aligned} \tag{2.7}$$

where

$$g_0{}^0 \equiv g_{11}, \quad g_0{}^1 \equiv g_{10}, \quad g_1{}^0 \equiv g_{01}, \quad g_1{}^1 \equiv g_{00} \tag{2.8}$$

An entity thus transformed under SL(2,C) is called a *two-component spinor*, and the corresponding representation is called *spinor representation*.

2.2. Spinors in Curved Space-Time

Consider a Riemannian manifold with connections. At each point define in some consistent manner a tangent two-dimensional complex space. At each such point we may define two-component spinors, which will be in general functions of the coordinates at each patch. Now in flat space-time every tensor is readily related to a two-component spinor through the Pauli matrices and the unit 2×2 matrix. Since at every point of the manifold space is locally flat, we conclude that this procedure can be done in curved space-time with some generalized Pauli and unit matrices such that when transforming the metric at a point to the Minkowskian metric, the transformed Pauli and unit matrices will correspond to the usual ones. This procedure is described now.

Let $\sigma^{\mu}_{AB'}$ be the generalized Pauli and unit matrices, where Greek letters correspond to space-time indexes-{0,1,2,3} and Latin capital letters denote spinorial indexes-{0,1}. These mixed quantities will behave like vectors under space-time change of coordinates and like two component spinors under $SL(2,C)$ group translations. They clearly preserve hermiticity as well,

$$\sigma^{\mu}_{AB'} = \overline{\sigma^{\mu}_{B'A}} = \bar{\sigma}^{\mu}_{BA'}. \tag{2.9}$$

The $\sigma^{\mu}_{AB'}$ are usually used for convenience and need not be calculated when spinors are used in general relativity. It is, however, important for our discussion to realize that once the metric tensor is known, the $\sigma^{\mu}_{AB'}$ can be found in some consistent way. This can be done for instance by the following procedure: (1) take a patch of the manifold with some coordinates, choose a point in the patch, and coordinate system in which the metric unifies with the Minkowskian metric at that point. At that point in the corresponding new coordinates the $\sigma^{\mu}_{AB'}$ can be chosen to be simply the normal flat space-time Pauli and unit matrices. Return to the original coordinates when the $\sigma^{\mu}_{AB'}$ transform as a vector. (2) Repeat (1) at all points in the patch. (3) Repeat (1) and (2) to all patches covering the manifold.

The relationship between the metric tensor and the $\sigma^{\mu}_{AB'}$ matrices, as in flat space-time, is given by

$$g_{\mu\nu}\sigma^{\mu}_{AB'}\sigma^{\nu}_{CD'} = \varepsilon_{AC}\varepsilon_{B'D'} \tag{2.10}$$

and

$$\varepsilon^{AC}\varepsilon^{B'D'}\sigma^{\mu}_{AB'}\sigma^{\nu}_{CD'} = g^{\mu\nu}, \tag{2.11}$$

where ε are totally skew-symmetric spinors with $\varepsilon_{01} \equiv 1$, called the Levi–Civita metric spinors. These spinors can be used in lowering and raising indexes as follows:

$$\begin{aligned} \xi^{A} &= \varepsilon^{AB}\xi_{B}, \qquad & \xi^{A'} &= \varepsilon^{A'B'}\xi_{B'}, \\ \xi_{A} &= \varepsilon_{BA}\xi^{B}, \qquad & \xi_{A'} &= \varepsilon_{B'A'}\xi^{B'}. \end{aligned} \tag{2.12}$$

Every tensor can be related to a spinor by means of "contracting" the tensor indices with the $\sigma^{\mu}_{AB'}$'s space-time indices. For example, the spinor equivalent to the tensor $T_{\alpha\beta}$ is given by

$$T_{AB'CD'} \equiv \sigma^{\alpha}_{AB'}\sigma^{\beta}_{CD'}T_{\alpha\beta}. \tag{2.13}$$

The transformation (2.13) is clearly reversible because of (2.10) and (2.11).

To relate spinors between two infinitesimally neighboring points and thus to define differentiation consistently, we need a law of parallel transplantation. This is done analogously to vector transplantation through spinor affine connections. If a spinor ξ_A is given at point P, we define the transplantation of ξ_A into $P + dP$ (Adler *et al.*, 1965).

$$\xi_A(P + dP) = \xi_A + \Gamma^{B}_{A\alpha}\xi_B. \tag{2.14}$$

Covariant differentiation is thus defined by

$$\nabla_\alpha \xi_A = \partial_\alpha \xi_A - \Gamma^B_{A\alpha} \xi_B. \tag{2.15}$$

The connections Γ are defined such that (1) The transplanted spinor transforms like a two-component spinor. (2) The covariant derivative transforms like a vector under change of coordinates. (3) $\nabla_\alpha \varepsilon_{AB} = \nabla_\alpha \varepsilon_{A'B'} = 0$. If in addition, (4) $\nabla_\alpha \sigma^\nu_{AB'} = 0$, then the spinor transplantation is called parallel transport or parallel displacement. The corresponding connections can thus be determined by the $\sigma^\mu_{AB'}$ and are called Riemannian spinor affine connection.

After this brief introduction to spinor algebra, and before discussing Carmeli's equations, some special spinors and identities are introduced.

2.3. The Curvature Spinor and Some Convenient Identities

Let us hereafter confine ourselves to only Riemannian spinor connections until otherwise specified. In analogy to Riemannian curvature we define spinorial curvature by the nonintegrability condition of (2.14) or equivalently by the commutator

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) \xi_A = F^B{}_{A\alpha\beta} \xi_B. \tag{2.16}$$

Now

$$\begin{aligned}
\nabla_\beta \nabla_\alpha \xi_A &= \partial_\beta \nabla_\alpha \xi_A - \Gamma^\gamma_{\alpha\beta} \nabla_\gamma \xi_A - \Gamma^B_{A\beta} \nabla_\alpha \xi_B \\
&= \partial_\beta \partial_\alpha \xi_A - \partial_\beta \Gamma^B_{A\alpha} \xi_B - \Gamma^\gamma_{\alpha\beta} \nabla_\gamma \xi_A - \Gamma^B_{A\beta} \partial_\alpha \xi_B + \Gamma^B_{A\beta} \Gamma^C_{B\alpha} \xi_C \\
&= \left(\partial_\beta \partial_\alpha \xi_A - \Gamma^\gamma_{\alpha\beta} \nabla_\gamma \xi_A - \Gamma^\beta_{A\beta} \partial_\alpha \xi_B - \Gamma^B_{A\alpha} \partial_\beta \xi_B\right) \\
&\quad + \left(\Gamma^B_{A\beta} \Gamma^C_{B\alpha} \xi_C - \left(\partial_\beta \Gamma^\beta_{A\alpha}\right) \xi_B\right)
\end{aligned} \tag{2.17}$$

The expression in the first brackets of the above equation is clearly symmetric and will therefore not contribute to (2.16). We thus find that

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) \xi_A = \left(\Gamma^B_{A\alpha,\beta} - \Gamma^B_{A\beta,\alpha} + \Gamma^C_{A\alpha} \Gamma^B_{C\beta} - \Gamma^C_{A\beta} \Gamma^B_{C\alpha}\right) \xi_B = F^B{}_{A\alpha\beta} \xi_B \tag{2.18}$$

or

$$F^B{}_{A\alpha\beta} = \Gamma^B_{A\alpha,\beta} - \Gamma^B_{A\beta,\alpha} + \Gamma^C_{A\alpha} \Gamma^B_{C\beta} - \Gamma^C_{A\beta} \Gamma^B_{C\alpha}. \tag{2.19}$$

$F^B{}_{A\alpha\beta}$ is called the *curvature spinor*.

The relationship between the Riemannian curvature tensor and the curvature spinor can be found as follows: recall that by definition, $R^\rho{}_{\nu\alpha\beta} \xi_\rho =$

$(\nabla_\beta\nabla_\alpha - \nabla_\alpha\nabla_\beta)\xi_\nu$. Multiplying both sides of this equation by $\sigma^\nu_{AB'}$ we find,

$$\sigma^\nu_{AB'}(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)\xi_\nu = \sigma^\nu_{AB'}R^\rho{}_{\nu\beta\alpha}\xi_\rho = \sigma^\nu_{AB'}g^{\rho\mu}R_{\mu\nu\beta\alpha}\xi_\rho$$
$$= \sigma^\nu_{AB'}\sigma^\mu_{CD'}\sigma^{\rho CD'}R_{\mu\nu\beta\alpha}\xi_\rho = R_{CD'AB'\beta\alpha}\xi^{CD'}. \quad (2.20)$$

Since we are strictly dealing with Riemannian spinor connections, condition (4) of Section 2b follows identically. The left-hand side of (2.20) is thus just

$$(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)\xi_{AB'} = F^M_{A\alpha\beta}\xi_{MB'} + \bar{F}^{M'}_{B'\alpha\beta}\xi_{AM'}$$
$$= (\varepsilon_{D'B'}F_{CA\beta\alpha} + \varepsilon_{CA}\bar{F}_{D'B'\beta\alpha})\xi^{CD'} \quad (2.21)$$

Comparing (2.20) and (2.21), we find

$$R_{AB'CD'\alpha\beta} = \varepsilon_{B'D'}F_{AC\alpha\beta} + \varepsilon_{AC}\bar{F}_{B'D'\alpha\beta}. \quad (2.22)$$

It can, moreover, be shown that the curvature spinor has the following symmetry properties:

$$F_{AB\alpha\beta} = F_{BA\alpha\beta},$$
$$F_{AB\alpha\beta} = -F_{AB\beta\alpha}. \quad (2.23)$$

Multiplying (2.22) by $\varepsilon^{B'D'}$ and applying the above symmetry properties, one obtains

$$2F_{AC\alpha\beta} = R_{AC\alpha\beta} \equiv R_{AB'C}{}^{B'}_{\alpha\beta.} \quad (2.24)$$

By the correspondence between the tensorial curvature and the spinorial curvature, we find that the spinor curvature satisfies the Bianchi identities

$$\nabla_\alpha R_{\mu\nu\beta\gamma} + \nabla_\beta R_{\mu\nu\gamma\alpha} + \nabla_\gamma R_{\mu\nu\alpha\beta} = 0, \quad (2.25)$$

$$\Updownarrow$$

$$0 = \sigma^\mu_{AE'}\sigma^{\nu E'}_B(\nabla_\alpha R_{\mu\nu\beta\gamma} + \nabla_\beta R_{\mu\nu\gamma\alpha} + \nabla_\gamma R_{\mu\nu\alpha\beta})$$
$$= \nabla_\alpha R_{AB\beta\gamma} + \nabla_\beta R_{AB\gamma\alpha} + \nabla_\gamma R_{AB\alpha\beta}, \quad (2.26)$$

$$\Updownarrow$$

$$\nabla_\alpha F_{AB\beta\gamma} + \nabla_\beta F_{AB\gamma\alpha} + \nabla_\gamma F_{AB\alpha\beta} = 0. \quad (2.27)$$

The dual of the curvature spinor is defined by

$${}^*F_{AB}{}^{\mu\nu} \equiv \frac{1}{2}(-g)^{-\frac{1}{2}}\varepsilon^{\alpha\beta\mu\nu}F_{AB\alpha\beta}. \quad (2.28)$$

We can thus write the Bianchi identities as $\nabla_\nu{}^*F_{AB}{}^{\mu\nu} = 0$, which is just Eq. (1b) of Carmeli's equations.

3. CORRESPONDENCE BETWEEN CARMELI'S AND EINSTEIN'S EQUATIONS

We are now in a position to discuss Carmeli's equations in more detail. We first recall, by Eq. (2.11), that the metric tensor can be written in terms of the $\sigma^{\nu}_{AB'}$ and the $\sigma_{\nu}^{AB'}$. This is true, therefore, for the Christoffel symbols, the spinor connections and thus, for the curvature spinor and its covariant derivatives. Carmeli's equations can thus be considered as a set of partial differential equations of the third order in the variables $\sigma^{\nu}_{AB'}$ and the $\sigma_{\nu}^{AB'}$ and their partial derivatives. Assuming, furthermore, that they are soluble, then we can build the metric and so the entire Riemannian space. We will now find out what the Riemannian space is so constructed. This will enable us to find out the correspondence to Einstein's equations.

Proposition 1. *When restricted to Riemannian connections, every solution to Einstein's equations*

$$R_{\mu}{}^{\nu} - \frac{1}{2}\delta_{\mu}{}^{\nu} R = kT_{\mu}{}^{\nu}, \tag{3.1}$$

satisfies Carmeli's equations identically for an appropriate choice of $J_{AB}{}^{\alpha}$ in Eq. (1a).

Before proving this statement, it will be convenient to prove the following lemma.

Lemma 1.1. *Let $H_{\alpha\beta}{}^{\gamma}$ be a tensor defined by*

$$H_{\alpha\beta}{}^{\gamma} \equiv \sigma_{\alpha}^{AB'}\sigma_{\beta}^{CD'}\left(\varepsilon_{B'D'}J_{AC}{}^{\gamma} + \varepsilon_{AC}\bar{J}_{B'D'}{}^{\gamma}\right) \tag{3.2}$$

Then

$$\nabla_{\delta} F_{AB}{}^{\gamma\delta} = J_{AB}{}^{\gamma} \Leftrightarrow \nabla_{\delta} R_{\alpha\beta}{}^{\gamma\delta} = H_{\alpha\beta}{}^{\gamma}. \tag{3.3}$$

Proof: We first notice that

$$\nabla_{\delta} F_{AB}{}^{\gamma\delta} = J_{AB}{}^{\gamma} \Leftrightarrow \nabla_{\delta} \bar{F}_{A'B'}{}^{\gamma\delta} = \bar{J}_{A'B'}{}^{\gamma}, \tag{3.4}$$

Hence by (2.22)

$$\begin{aligned}\nabla_{\nu} R_{AB'CD'}{}^{\mu\nu} &= \varepsilon_{B'D'}\nabla_{\nu} F_{AC}{}^{\mu\nu} + \varepsilon_{AC}\nabla_{\nu}\bar{F}_{B'D'}{}^{\mu\nu}\\ &= \varepsilon_{B'D'}J_{AC}{}^{\mu} + \varepsilon_{AC}\bar{J}_{B'D'}{}^{\mu}.\end{aligned} \tag{3.5}$$

Conversely, if (3.5) holds, then multiplying its both sides by $\varepsilon^{B'D'}$ gives (la). Consequently,

$$\nabla_{\nu} F_{AB}{}^{\mu\nu} = J_{AB}{}^{\mu} \Leftrightarrow \nabla_{\nu} R_{AC'BD'}{}^{\mu\nu} = \varepsilon_{C'D'}J_{AB}{}^{\mu} + \varepsilon_{AB}\bar{J}_{C'D'}{}^{\mu}. \tag{3.6}$$

But since we are dealing with Riemannian connections, condition (4) of Section 2b is satisfied. Therefore

$$\nabla_\nu R_{\alpha\beta}{}^{\mu\nu} = H_{\alpha\beta}{}^{\mu} \Leftrightarrow \nabla_\nu R_{AC'BD'}{}^{\mu\nu} = \varepsilon_{C'D'} J_{AB}{}^{\mu} + \varepsilon_{AB} \bar{J}_{C'D'}{}^{\mu} \tag{3.7}$$

since $R_{\alpha\beta}{}^{\mu\nu} = \sigma_\alpha^{AB'}\sigma_\beta^{CD'} R_{AB'CD'}{}^{\mu\nu}$ and the transformation is reversible. Thus (3.6) and (3.7) give (3.3). This proves our lemma. □

Proof of Proposition 1: The major significance of the restriction to Riemannian connections is that Bianchi identities (2.25) and (1b) are satisfied. Multiplying (2.25) by $g^{\alpha\mu}$ gives

$$\nabla_\mu R_{\beta\gamma\nu}{}^{\mu} = \nabla_\gamma R_{\beta\nu} - \nabla_\beta R_{\gamma\nu}. \tag{3.8}$$

We can write the Einstein equations (3.1) as

$$R_{\alpha\beta} = k(T_{\alpha\beta} - g_{\alpha\beta} T), \tag{3.9}$$

were $T \equiv g^{\mu\nu} T_{\mu\nu}$. But if Einstein's equations (3.9) are satisfied, then by (3.8) we have

$$\nabla_\mu R_{\beta\gamma\nu}{}^{\mu} = k(\nabla_\gamma W_{\beta\nu} - \nabla_\beta W_{\gamma\nu}) \tag{3.10}$$

where

$$W_{\alpha\beta} \equiv T_{\alpha\beta} - g_{\alpha\beta} T. \tag{3.11}$$

Now let

$$\nabla_\mu R_{\alpha\beta}{}^{\gamma\mu} = H_{\alpha\beta}{}^{\gamma} \tag{3.12}$$

where $H_{\alpha\beta}{}^{\gamma}$ is given by Eq. (3.2). Then (3.10) gives

$$H_{\alpha\beta}{}^{\gamma} \equiv \sigma_\alpha^{AB'}\sigma_\beta^{CD'}\left(\varepsilon_{B'D'} J_{AC}{}^{\gamma} + \varepsilon_{AC} J_{B'D'}{}^{\gamma}\right) = k\left(\nabla_\beta W_\alpha{}^{\gamma} - \nabla_\alpha W_\beta{}^{\gamma}\right) \tag{3.13}$$

$$\Updownarrow$$

$$\begin{aligned} \varepsilon_{B'D'} J_{AC}{}^{\gamma} + \varepsilon_{AC} J_{B'D'}{}^{\gamma} &= k\sigma^\alpha_{AB'}\sigma^\beta_{CD'}\left(\nabla_\beta W_\alpha{}^{\gamma} - \nabla_\alpha W_\beta{}^{\gamma}\right) \\ &= k\left(\nabla_{CD'} W_{AB'}{}^{\gamma} - \nabla_{AB'} W_{CD'}{}^{\gamma}\right) \end{aligned} \tag{3.14}$$

$$\Updownarrow$$

$$\begin{aligned} J_{AC}{}^{\gamma} &= \frac{1}{2} k \varepsilon^{B'D'}\left(\nabla_{CD'} W_{AB'}{}^{\gamma} - \nabla_{AB'} W_{CD'}{}^{\gamma}\right) \\ &= \frac{1}{2} k\left(\nabla_C{}^{B'} W_{AB'}{}^{\gamma} + \nabla_A{}^{B'} W_{CB'}{}^{\gamma}\right). \end{aligned} \tag{3.15}$$

Let $J_{AB}{}^{\gamma}$ be given as in (3.15). If $R_{\alpha\beta\gamma\delta}$ is the curvature tensor built from a solution to Einstein's equations, then, as shown, (3.10) is satisfied. But if (3.15) is satisfied, then (3.13) is satisfied, and if (3.10) and (3.13) are satisfied, then (3.12) is satisfied, so that by Lemma 1.1, Eq. (1a) is satisfied, proving our proposition. □

We conclude from Proposition 1 that Carmeli's equations are more general than Einstein's equations. It is now desired to check the extent of this generality when ${J_{AB}}^{\gamma}$ is given by (3.15). To this end, let $R_{\alpha\beta\gamma\delta}$ correspond to a solution to Einstein's equations and let $\tilde{R}_{\alpha\beta\gamma\delta}$ correspond to a solution to Carmeli's equations with ${J_{AB}}^{\gamma}$ given by (3.15). Then by Proposition 1, $R_{\alpha\beta\gamma\delta}$ correspond to a solution to Carmeli's equations, so that by Lemma 1.1 and Eq. (3.2), both $R_{\alpha\beta\gamma\delta}$ and $\tilde{R}_{\alpha\beta\gamma\delta}$ satisfy Eq. (3.12). Thus

$$\nabla_{\delta} {R_{\alpha\beta}}^{\gamma\delta} = \nabla_{\delta} {\tilde{R}_{\alpha\beta}}^{\gamma\delta} = {H_{\alpha\beta}}^{\gamma}. \tag{3.16}$$

Let

$$A_{\alpha\beta\gamma\delta} \equiv \tilde{R}_{\alpha\beta\gamma\delta} - R_{\alpha\beta\gamma\delta}. \tag{3.17}$$

Then by Eq. (3.16) $A_{\alpha\beta\gamma\delta}$ satisfies

$$\nabla_{\delta} {A_{\alpha\beta\gamma}}^{\delta} = 0. \tag{3.18}$$

It is also clear that $A_{\alpha\beta\gamma\delta}$ has the same symmetry properties as $\tilde{R}_{\alpha\beta\gamma\delta}$ and $R_{\alpha\beta\gamma\delta}$. Moreover, since both $\tilde{R}_{\alpha\beta\gamma\delta}$ and $R_{\alpha\beta\gamma\delta}$ satisfy Bianchi identities then so does $A_{\alpha\beta\gamma\delta}$. Thus by Lemma 1.1 and Eq. (3.18) $A_{\alpha\beta\gamma\delta}$ corresponds to a solution to the homogeneous Carmeli's equations, namely Eq. (1a), (1b) with ${J_{AB}}^{\alpha} = 0$.

Note that since $A_{\alpha\beta\gamma\delta}$ has the same symmetry properties as the curvature tensor, and since it satisfies Bianchi's identities, then (3.18) can be written through (3.8) as

$$\nabla_{\mu} {A_{\beta\gamma\nu}}^{\mu} = \nabla_{\gamma} A_{\beta\nu} - \nabla_{\beta} A_{\gamma\nu} = 0, \tag{3.19}$$

where

$$A_{\alpha\beta} = {A_{\alpha\mu\beta}}^{\mu} = g^{\mu\nu} A_{\alpha\mu\beta\nu}. \tag{3.20}$$

The contraction of (3.19) through β, ν gives

$$2\nabla_{\mu} {A_{\gamma}}^{\mu} = \partial_{\gamma} A \tag{3.21}$$

and

$$\nabla_{\mu} {A_{\gamma}}^{\mu} = \partial_{\gamma} A. \tag{3.22}$$

We thus conclude

$$\begin{gathered} \partial A = 0 \\ \Downarrow \\ A = \text{const.} \end{gathered} \tag{3.23}$$

Now contracting (3.17), we get

$$\tilde{R}_{\alpha\beta} = R_{\alpha\beta} + A_{\alpha\beta}. \tag{3.24}$$

So we see that a solution to the Carmeli equations with the corresponding "current density" is equivalent to a solution to a generalized Einstein equations of the sort

$$R_{\alpha\beta} = k(T_{\alpha\beta} - g_{\alpha\beta}T) + A_{\alpha\beta}, \tag{3.25}$$

where $A_{\alpha\beta}$ is the Ricci tensor that corresponds to a solution to the homogeneous Carmeli equations. Recall that the Einstein field equations in empty space with a cosmological constant can be written as $R_{\alpha\beta} = g_{\alpha\beta}\Lambda$, and since, by definition of Riemannian space, the metric is a constant field with respect to covariant differentiation, we arrive at the conclusion that such a Ricci tensor must satisfy Eq. (3.19). Furthermore, by Lemma 1.1 we conclude that $R_{\alpha\beta}$ also corresponds to a solution to the homogeneous Carmeli equations. There exists, therefore, a solution to the homogeneous Carmeli equations such that the corresponding Ricci tensor $A_{\alpha\beta}$ satisfies $A_{\alpha\beta} = g_{\alpha\beta}\Lambda$. Substituting this in Eq. (3.25), we see that solutions to the Einstein equations with a cosmological constant are also included in solutions to Carmeli's equations.

4. CONCLUDING REMARKS

The Carmeli equations, as we have seen, are more general than Einstein's equations. They, moreover, contain all solutions to Einstein's equations even with a cosmological constant. It is worthwhile therefore exploring them furthermore. In particular, as shown in the last section, their homogeneous solutions play an important role.

It is interesting to note, and may be shown in a sequel paper, that the Riemannian spinor connections are readily in the form of $SL(2,\mathrm{C})$ gauge potentials, with the curvature spinor as the gauge field. The gauge field equations thus formed will clearly be the Carmeli equations. If, moreover, the connections are not specified, then the Carmeli equations will be a general $SL(2,\mathrm{C})$ gauge field equations.

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